



Fracture Mechanics

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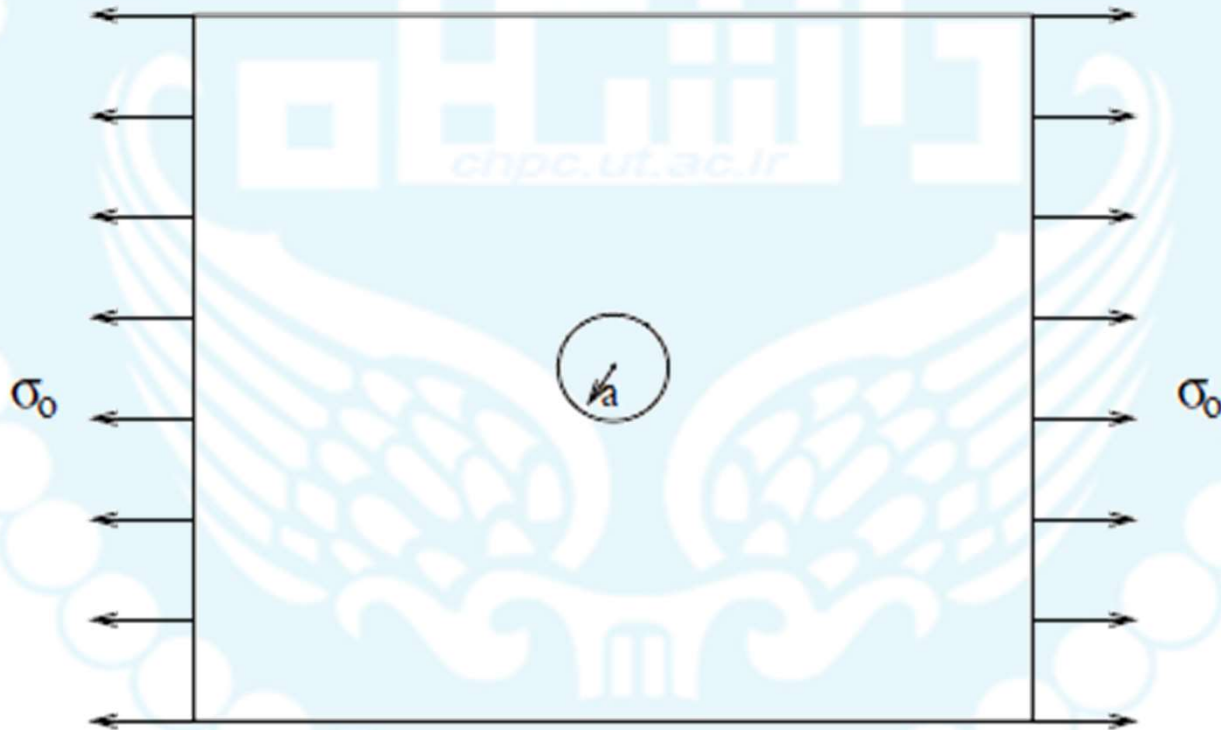
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The background features a large, faint, light blue watermark of the University of Mohammedi Seal. The seal is circular with a scalloped outer edge. Inside the seal, there is a central emblem with Arabic calligraphy. The words "UNIVERSITY OF MOHAMMEDI" are written in a circular path around the central emblem, with each letter in its own small circle.

Infinite Tensile Plate with a Circular Hole

Infinite tensile plate with a circular hole

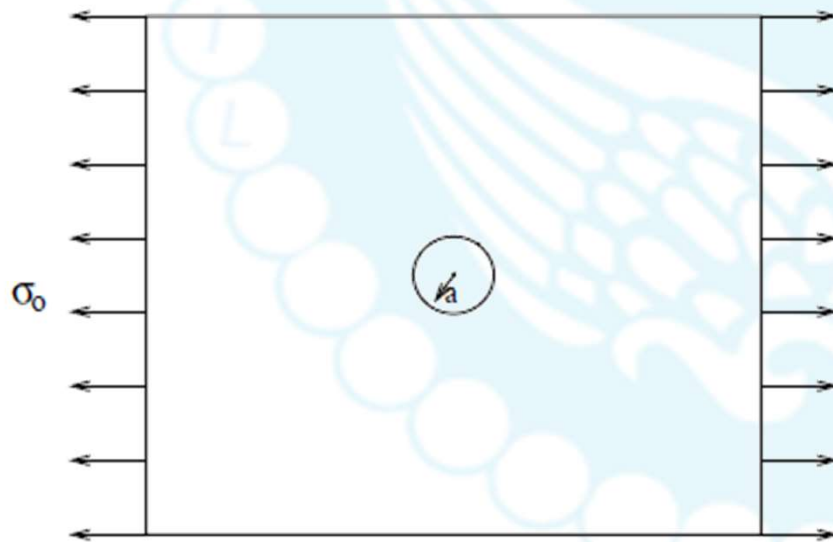
In 1898, Kirsch analysed the problem of an infinite plate with a circular hole under uniform tensile stress,



Infinite tensile plate with a circular hole

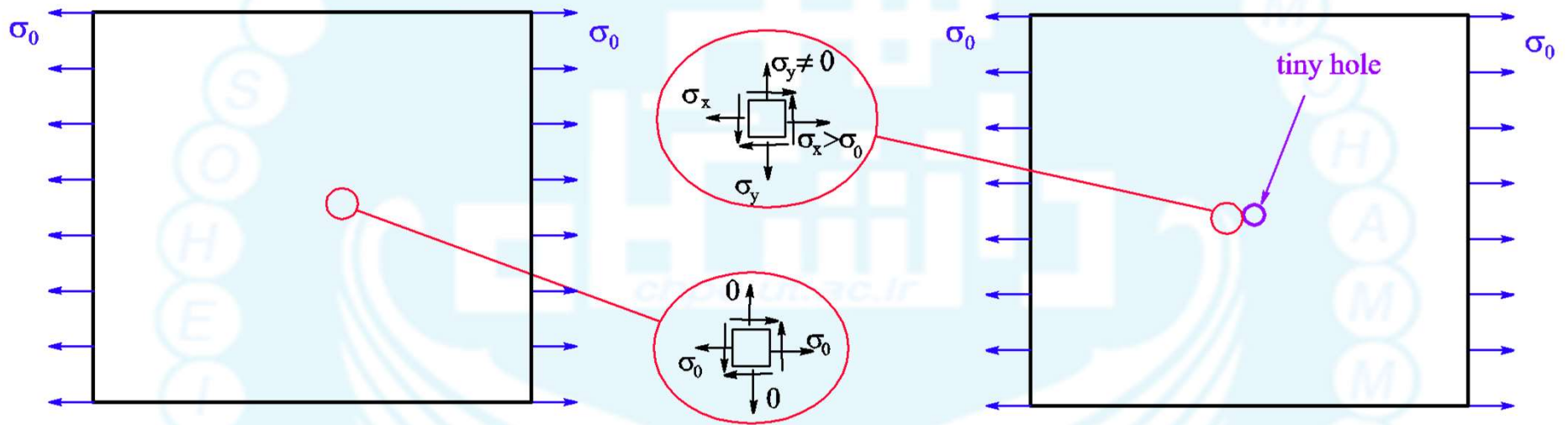


Uniform stress
Uni-directional stress



Non-uniform stress
Multi-axial stress
Independent of hole size

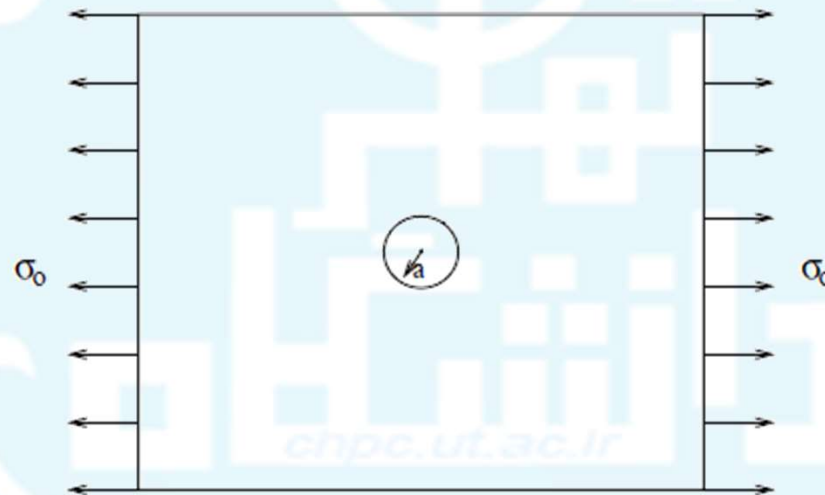
Infinite tensile plate with a circular hole



Uniform stress
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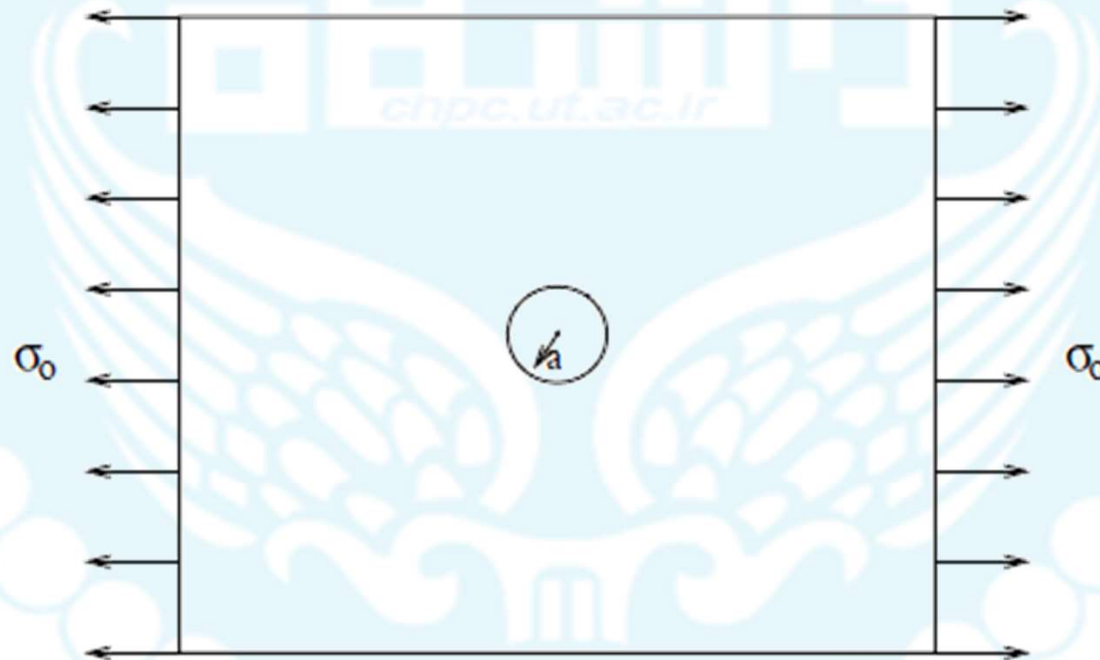
Infinite tensile plate with a circular hole



Problem	Coordinate System	Real/Complex	Solution	Date
Circular Hole	Polar	Real	Kirsh	1898
Elliptical Hole	Curvilinear	Complex	Inglis	1913
Crack	Cartesian	Complex	Westergaard	1939
V Notch	Polar	Complex	Willimas	1952
Dissimilar Materials	Polar	Complex	Williams	1959
Anisotropic Materials	Cartesian	Complex	Sih	1965

Infinite tensile plate with a circular hole

The problem is difficult to analyse in a single coordinate system. The boundary conditions around the circular hole can naturally be expressed in polar coordinate systems, whereas the far field Cartesian boundary conditions are better expressed in the xy coordinate system.



First we select a stress function which satisfies the biharmonic Equation, and the far-field boundary conditions.

$$\nabla^2 (\nabla^2 \Phi) = 0$$

From St Venant principle, away from the hole, the boundary conditions are given by:

$$\sigma_{xx} = \sigma_0$$

$$\sigma_{yy} = \sigma_{xy} = 0$$

Recalling

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2}$$

would suggest a stress function Φ of the form

$$\Phi = \sigma_0 y^2$$

Alternatively, the presence of the circular hole would suggest a polar representation of Φ .

Thus, substituting $y = r \sin \theta$ would result in

$$\Phi = \sigma_0 r^2 \sin^2 \theta$$

Since

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

we could simplify the stress function into

$$\Phi = f(r) \cos 2\theta$$

Substituting this function into the biharmonic equation yields
(note that the $\cos 2\theta$ term is dropped)

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} \right) = 0$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{4}{r^2} \right) \left(\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} - \frac{4f}{r^2} \right) = 0$$

The general solution of this ordinary linear fourth order differential equation is

$$f(r) = Ar^2 + Br^4 + C \frac{1}{r^2} + D$$

thus the stress function becomes

$$\Phi = \left(Ar^2 + Br^4 + C \frac{1}{r^2} + D \right) \cos 2\theta$$

Next, we must determine the constants A, B, C, and D.

The stresses are then given by

$$\sigma_{rr} = \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} = - \left(2A + \frac{6C}{r^4} + \frac{4D}{r^2} \right) \cos 2\theta$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2} = \left(2A + 12Br^2 + \frac{6C}{r^4} \right) \cos 2\theta$$

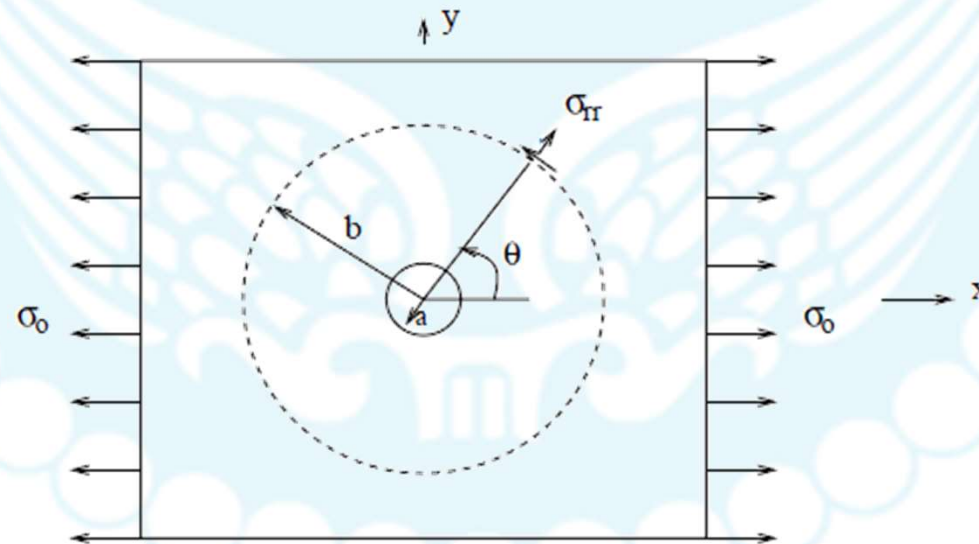
$$\sigma_{r\theta} = - \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Phi}{\partial \theta} \right) = \left(2A + 6Br^2 - \frac{6C}{r^4} - \frac{2D}{r^2} \right) \sin 2\theta$$

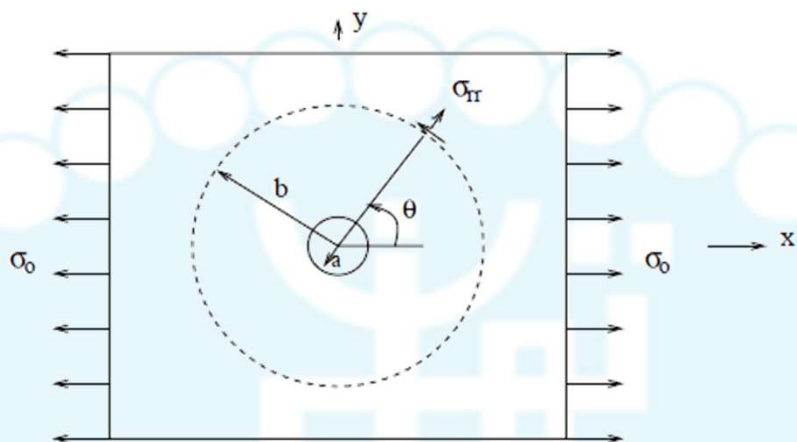
Next, the four constants of integration are solved by applying the boundary conditions.

Two sets of boundary conditions can be identified:

1. Outer boundaries:

Around an infinitely large circle of radius b inside a plate subjected to uniform stress σ_0 , the stresses in polar coordinates are obtained from the Strength of Materials





$$\begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{r\theta} & \sigma_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \sigma_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}^T$$

yielding $\left[\sin^2 \theta = \frac{1}{2} \sin 2\theta, \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) \right]$

$$(\sigma_{rr})_{r=b} = \sigma_0 \cos^2 \theta = \frac{1}{2} \sigma_0 (1 + \cos 2\theta)$$

$$(\sigma_{r\theta})_{r=b} = \frac{1}{2} \sigma_0 \sin 2\theta$$

$$(\sigma_{\theta\theta})_{r=b} = \frac{1}{2} \sigma_0 (1 - \cos 2\theta)$$

For reasons which will become apparent later, it is more convenient to decompose the state of stress into state *I* and *II*:

$$(\sigma_{rr})_{r=b}^I = \frac{1}{2} \sigma_0$$

$$(\sigma_{r\theta})_{r=b}^I = 0$$

$$(\sigma_{rr})_{r=b}^{II} = \frac{1}{2} \sigma_0 \cos 2\theta$$

$$(\sigma_{r\theta})_{r=b}^{II} = \frac{1}{2} \sigma_0 \sin 2\theta$$

State I corresponds to a thick cylinder with external pressure applied on $r=b$ and of magnitude $\sigma_0/2$.

Hence, only the last two equations will provide us with boundary conditions.

2. Around the hole:

The stresses should be equal to zero:

$$(\sigma_{rr})_{r=a} = 0$$

$$(\sigma_{r\theta})_{r=a} = 0$$

Upon substitution, the four boundary conditions become

$$-\left(2A + \frac{6C}{b^4} + \frac{4D}{b^2}\right) = \frac{1}{2}\sigma_0$$

$$\left(2A + 6Bb^2 - \frac{6C}{b^4} - \frac{2D}{b^2}\right) = \frac{1}{2}\sigma_0$$

$$-\left(2A + \frac{6C}{a^4} + \frac{4D}{a^2}\right) = 0$$

$$\left(2A + 6Ba^2 - \frac{6C}{a^4} - \frac{2D}{a^2}\right) = 0$$

Solving for the four unknowns, and taking $a/b=0$ (i.e. an infinite plate), we obtain:

$$A = -\frac{1}{4}\sigma_0, \quad B = 0, \quad C = -\frac{a^4}{4}\sigma_0, \quad D = \frac{a^2}{2}\sigma_0$$

To this solution, we must superimpose the one of a thick cylinder subjected to a uniform radial traction $\sigma_0/2$ on the outer surface, and with b much greater than a .

These stresses are obtained from Strength of Materials yielding for this problem (careful about the sign).

$$\sigma_{rr} = \frac{1}{2}\sigma_0 \left(1 - \frac{a^2}{r^2} \right)$$

$$\sigma_{\theta\theta} = \frac{1}{2}\sigma_0 \left(1 + \frac{a^2}{r^2} \right)$$

Substitution of A, B, C and D:

$$\sigma_{rr} = \frac{1}{2} \sigma_0 \left(1 - \frac{a^2}{r^2} \right) + \frac{1}{2} \sigma_0 \cos 2\theta \left(1 + 3 \frac{a^4}{r^4} - 4 \frac{a^2}{r^2} \right)$$

$$\sigma_{\theta\theta} = \frac{1}{2} \sigma_0 \left(1 + \frac{a^2}{r^2} \right) - \frac{1}{2} \sigma_0 \cos 2\theta \left(1 + 3 \frac{a^4}{r^4} \right)$$

$$\sigma_{r\theta} = -\frac{1}{2} \sigma_0 \sin 2\theta \left(1 - 3 \frac{a^4}{r^4} + 2 \frac{a^2}{r^2} \right)$$

We observe that as $r \rightarrow \infty$, both σ_{rr} and $\sigma_{r\theta}$ are equal to the values given for outer boundaries.

Alternatively, at the edge of the hole when $r=a$ we obtain

$$(\sigma_{rr})_{r=a} = 0$$

$$(\sigma_{r\theta})_{r=a} = 0$$

$$(\sigma_{\theta\theta})_{r=a} = \sigma_0 (1 - 2 \cos 2\theta)$$

which for $\theta = \pi/2$ and $\theta = 3\pi/2$ gives a stress concentration factor (SCF) of 3. For $\theta = 0$ and $\theta = \pi$, $\sigma_{\theta\theta} = -\sigma_0$ (**compressive**).

2nd approach

The problem is assumed to be combination of two problems:

- 1- an intact infinite tensile plate
- 2- a plate with induced circular hole



For the intact infinite tensile plate, a uniform stress exists and the following stress function can be used:

$$\phi_1 = \frac{\sigma_0}{2} y^2$$

$$\sigma_{xx} = \sigma_0$$

$$\sigma_{yy} = \sigma_{xy} = 0$$



Or in polar coordinates:

$$\phi_1 = \frac{\sigma_0}{4} r^2 (1 - \cos 2\theta)$$

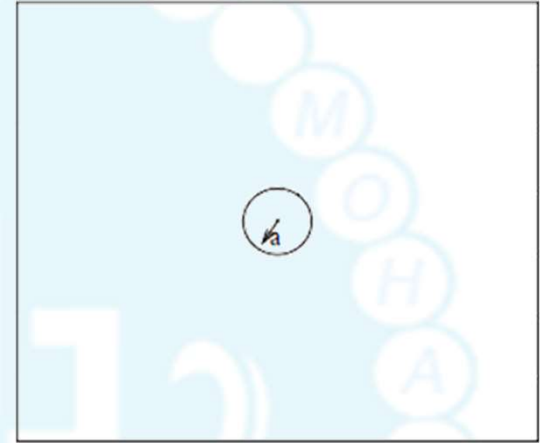
$$\sigma_{rr}^{(1)} = \frac{\sigma_0}{2} (1 + \cos 2\theta)$$

$$\sigma_{\theta\theta}^{(1)} = \frac{\sigma_0}{2} (1 - \cos 2\theta)$$

$$\sigma_{r\theta}^{(1)} = -\frac{\sigma_0}{2} \sin 2\theta$$

Now, if a circular hole (with radius a) is placed inside a uniform plate, the internal boundary must satisfy (for all values of θ):

$$\sigma_{rr} = \sigma_{r\theta} = 0 \quad \text{at } r = a$$



It can be shown that the following stress functions can produce stress fields around the hole opposite to values obtained from part (1):

$$\phi_2 = \frac{\sigma_0}{2} a^2 \left[-\ln r + \left(1 - \frac{1}{2} \left(\frac{a}{r} \right)^2 \right) \cos 2\theta \right]$$

$$\phi_2 = \frac{\sigma_0}{2} a^2 \left[-\ln r + \left(1 - \frac{1}{2} \left(\frac{a}{r} \right)^2 \right) \cos 2\theta \right]$$

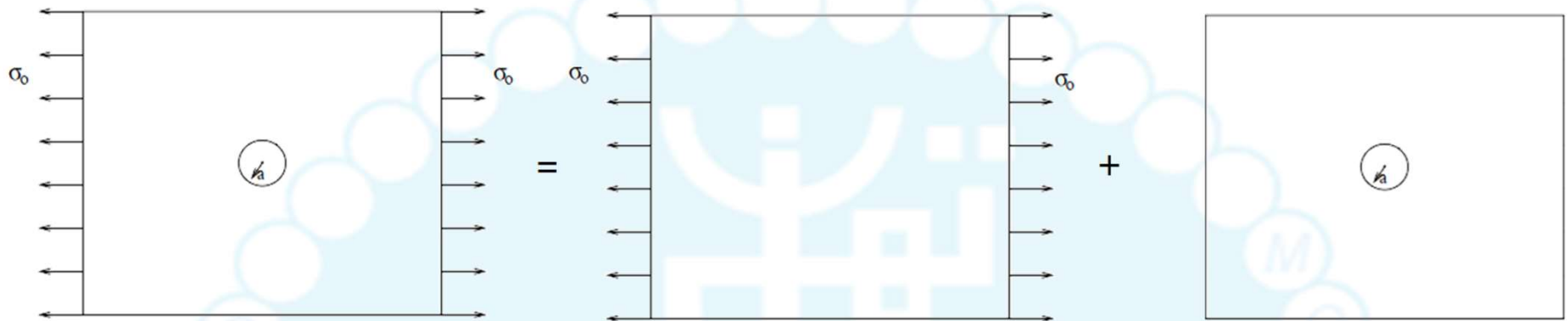


After differentiation:

$$\sigma_{rr}^{(2)} = \frac{\sigma_0 a^2}{2r^2} \left[-1 + \left(-4 + 3 \left(\frac{a}{r} \right)^2 \right) \cos 2\theta \right]$$

$$\sigma_{\theta\theta}^{(2)} = \frac{\sigma_0 a^2}{2r^2} \left[1 - 3 \left(\frac{a}{r} \right)^2 \cos 2\theta \right]$$

$$\sigma_{r\theta}^{(2)} = \frac{\sigma_0 a^2}{2r^2} \left[-2 + 3 \left(\frac{a}{r} \right)^2 \right] \sin 2\theta$$

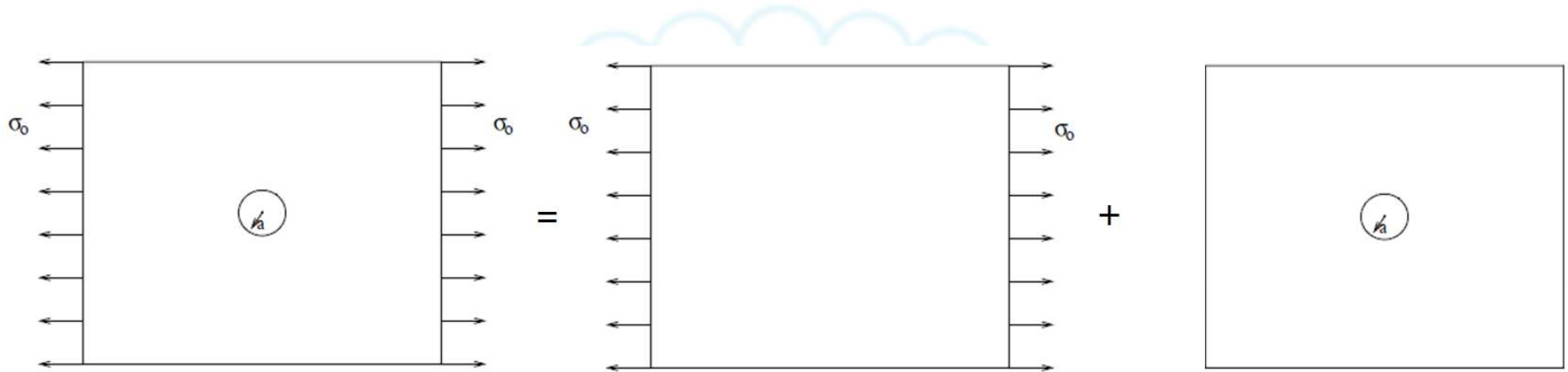


These stress functions must satisfy:

1- On the hole boundary ($r=a$):

$$\sigma_{rr}^{(2)}(r = a) = -\frac{\sigma_0}{2}(1 + \cos 2\theta) \quad \left[= -\sigma_{rr}^{(1)}(r = a) \right]$$

$$\sigma_{r\theta}^{(2)}(r = a) = \frac{\sigma_0}{2}\sin 2\theta \quad \left[= -\sigma_{r\theta}^{(1)}(r = a) \right]$$



2- On the far boundary ($r \rightarrow \infty$):

$$\sigma_{rr}^{(2)} = \frac{\sigma_0}{2} \left[-\frac{a^2}{r^2} + \left(3\frac{a^4}{r^4} - 4\frac{a^2}{r^2} \right) \cos 2\theta \right] \approx 0 \quad \text{as } r \rightarrow \infty$$

$$\sigma_{\theta\theta}^{(2)} = \frac{\sigma_0}{2} \left[\frac{a^2}{r^2} - 3\frac{a^4}{r^4} \cos 2\theta \right] \approx 0 \quad \text{as } r \rightarrow \infty$$

$$\sigma_{\theta\theta}^{(2)} = \frac{\sigma_0}{2} \left[\left(3\frac{a^4}{r^4} - 2\frac{a^2}{r^2} \right) \sin 2\theta \right] \approx 0 \quad \text{as } r \rightarrow \infty$$

The final solution is obtained from the sum of two states (for all points (r, θ)):

$$\sigma_{rr} = \sigma_{rr}^{(1)} + \sigma_{rr}^{(2)} = \frac{\sigma_0}{2} \left\{ \left(1 - \left(\frac{a}{r} \right)^2 \right) \left(1 + \left[1 - 3 \left(\frac{a}{r} \right)^2 \right] \cos 2\theta \right) \right\}$$

$$\sigma_{\theta\theta} = \sigma_{\theta\theta}^{(1)} + \sigma_{\theta\theta}^{(2)} = \frac{\sigma_0}{2} \left\{ \left(1 + \left(\frac{a}{r} \right)^2 \right) - \left[1 + 3 \left(\frac{a}{r} \right)^2 \right] \cos 2\theta \right\}$$

$$\sigma_{r\theta} = \sigma_{r\theta}^{(1)} + \sigma_{r\theta}^{(2)} = -\frac{\sigma_0}{2} \left\{ \left(1 + 3 \left(\frac{a}{r} \right)^2 \right) \left(1 - \left(\frac{a}{r} \right)^2 \right) \sin 2\theta \right\}$$

The stress state along the y -axis can now be computed as:

$$y\text{-axis} : \left(r, \theta = \pm \frac{\pi}{2} \right)$$

$$\sigma_{rr} = \sigma_{yy} = \frac{\sigma_0}{2} \left\{ 3 \left(\frac{a}{y} \right)^2 \left[1 - \left(\frac{a}{y} \right)^2 \right] \right\}$$

$$\sigma_{\theta\theta} = \sigma_{xx} = \frac{\sigma_0}{2} \left\{ 2 + \left(\frac{a}{y} \right)^2 + \left(\frac{a}{y} \right)^4 \right\}$$

$$\sigma_{r\theta} = \sigma_{xy} = 0$$

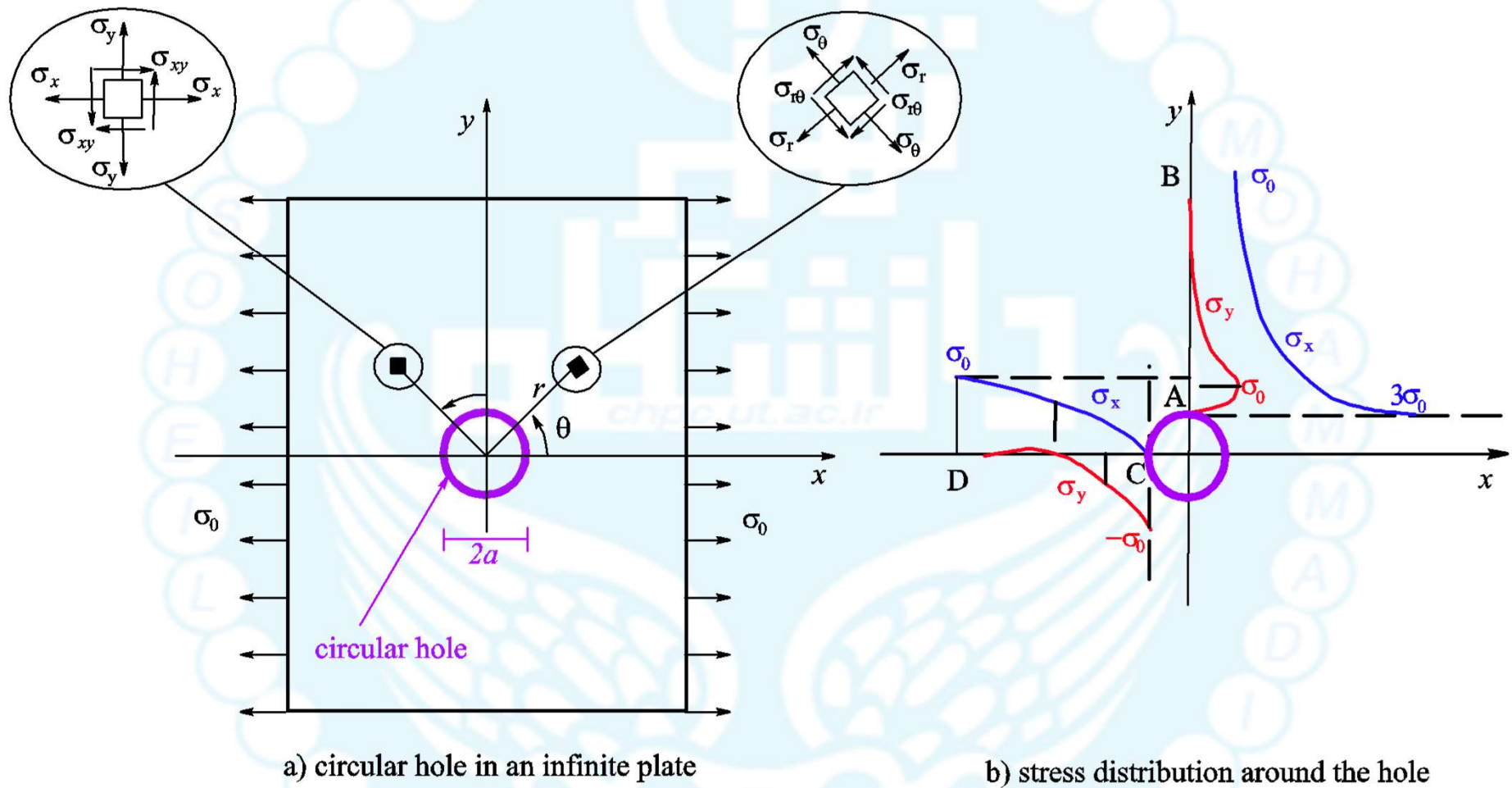
The stress state along the x -axis can now be computed as:

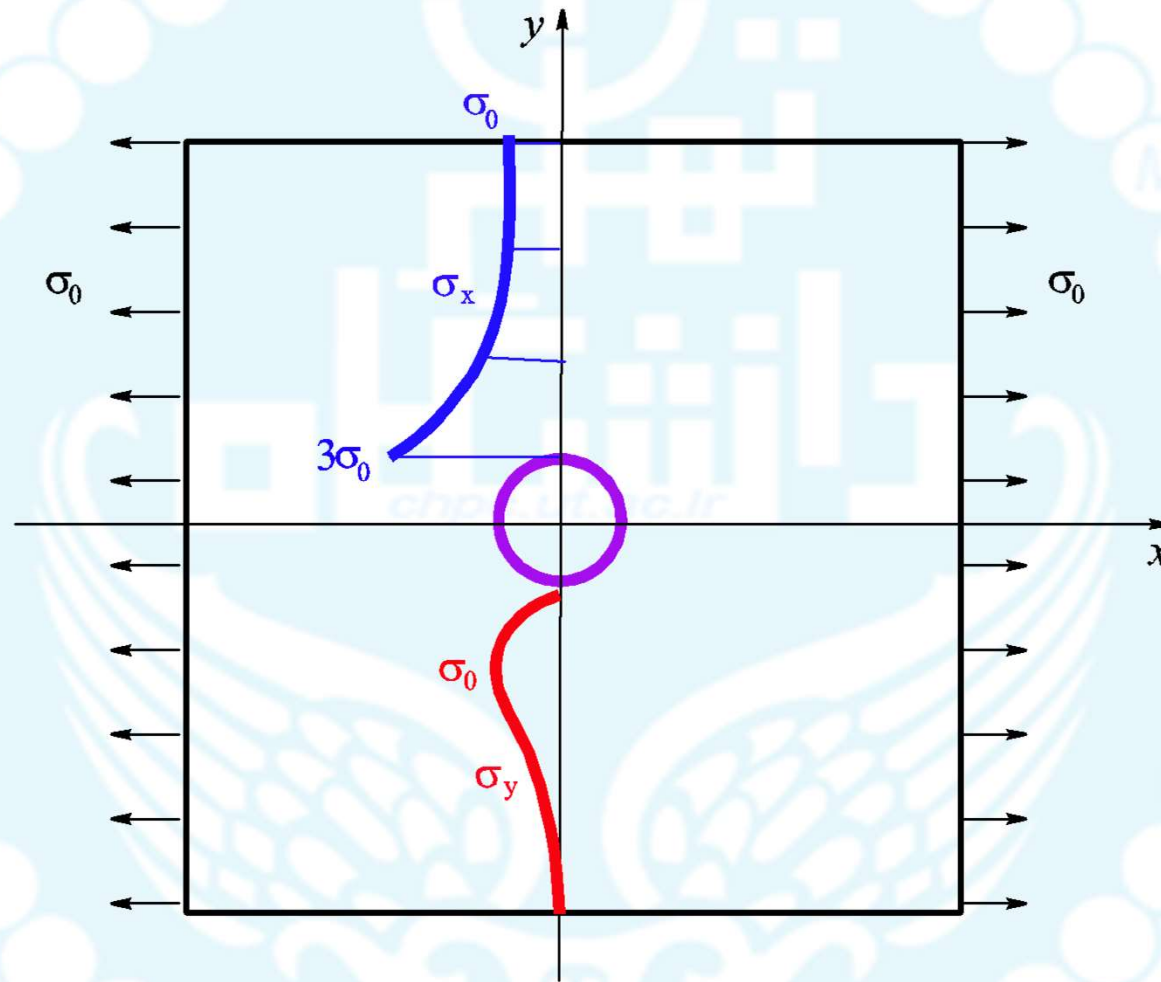
x -axis : $(r, \theta = 0)$

$$\sigma_{rr} = \sigma_{xx} = \frac{\sigma_0}{2} \left\{ 2 - 5 \left(\frac{a}{x} \right)^2 + 3 \left(\frac{a}{x} \right)^4 \right\}$$

$$\sigma_{\theta\theta} = \sigma_{yy} = \frac{\sigma_0}{2} \left\{ \left(\frac{a}{x} \right)^2 - 3 \left(\frac{a}{x} \right)^4 \right\}$$

$$\sigma_{r\theta} = \sigma_{xy} = 0$$





It is noticed that by inclusion of a circular hole into an infinite tensile plate (defected tensile plate):

1. the uniform stress state is changed to non-uniform state with a stress concentration factor (SCF) of 3.
2. For a uniform uni-directional applied stress, the plate experiences a multi-axial stress state.
3. Change of stress sign is observed. Some parts of the defected tensile plate become compressive.
4. As a result, potential buckling modes exists in the defected tensile plate.
5. Existence of the non-uniform and tensile/compressive stress states may increase the potential fatigue failure modes.
6. The conclusions does not depend to the hole size. As a result, even a tiny micro hole may triple the maximum tensile stress and generate a compressive stress of the order of applied stress.

HW P2-C02#1:

Draw the stress contours for an infinite tensile plate with a central hole:

- 1- in the (x,y) Cartesian system
- 2- in the (r,θ) polar system



